

MICZ-Kepler Problems in All Dimensions

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Abstract

The Kepler problem is a physical problem about two bodies which attract each other by a force proportional to the inverse square of the distance. The MICZ-Kepler problems are its natural cousins and have been previously generalized from dimension three to dimension five. In this paper, we construct and analyze the (quantum) MICZ-Kepler problems in all dimensions higher than two.

1 Introduction

The Kepler problem is the physics problem about two bodies which attract each other by a force proportional to the inverse square of the distance. By solving this problem in classical mechanics, Newton gave a satisfactory explanation for Kepler's laws for the planetary motion. The Kepler problem plays a significant role in the development of quantum mechanics, too; in fact, the solution of this problem in the Schrödinger's wave mechanics firmly puts the Schrödinger equation right at the center of quantum mechanics.

After more than three centuries, the Kepler problem still plays an important role in mathematics and physics. There has been a continuous interest in this problem; in particular, in the last three decades we have witnessed an explosion of its interactions with quantum mechanics, celestial mechanics and mathematics. For a recent comprehensive treatment of the Kepler problem, the interested readers may consult Ref. [1].

The MICZ-Kepler problems are natural cousins of the Kepler problem, and they were independently discovered by McIntosh-Cisneros [2] and Zwanziger [3] more than thirty years ago. Roughly speaking, a MICZ-Kepler problem is the Kepler problem in the case when the nucleus of a hypothetical hydrogen atom also carries a magnetic charge. These generalized problems share the following characteristic beauty with the Kepler problem: the existence of the Runge-Lenz vector and the dynamical $\text{Spin}(4)$ symmetry for the bound states; therefore, they

provide a rich family of examples for the exploration of extra hidden dynamic symmetry.

The hamiltonian of a MICZ-Kepler problem is constructed from that of the Kepler problem by adding the vector potential of a Dirac monopole and a repulsive centrifugal potential; explicitly, we have

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 + \frac{\mu^2}{2mr^2} - \frac{e^2}{r} \quad (1)$$

where \vec{p} is the canonical momentum of the electron, \vec{A} is the vector potential of a Dirac monopole, r is the distance from the electron to the hydrogen nucleus, m is the (reduced) mass of the electron, e is the fundamental unit of the electric charge, and μ is the magnetic charge of the Dirac monopole measured in unit $\frac{c}{e}$, i.e., μ_e^c is the magnetic charge of the Dirac monopole, here c is the speed of light in vacuum¹.

Quantum mechanically, via rescaling

$$r \rightarrow \frac{\hbar^2}{me^2}r, \quad \mu \rightarrow \hbar\mu,$$

we arrive at the following hamiltonian operator:

$$\hat{H} = \frac{me^4}{\hbar^2} \left(-\frac{1}{2}\Delta_A + \frac{\mu^2}{2r^2} - \frac{1}{r} \right) := \frac{me^4}{\hbar^2} \hat{h}$$

where²

$$\hat{h} = -\frac{1}{2}\Delta_A + \frac{\mu^2}{2r^2} - \frac{1}{r}. \quad (2)$$

Here Δ_A is the Laplace operator twisted by the gauge potential A of a Dirac monopole, and $\mu = 0, \pm\frac{1}{2}, \pm 1, \dots$ is the magnetic charge of the Dirac monopole measured in terms of the fundamental unit³. Locally, with a gauge chosen, we have

$$\Delta_A = \nabla_a \nabla_a$$

where the repeated index a is summed up. Here ∇_a is the a -th covariant partial derivative and is written as $\partial_a + iA_a$ by physicists with $\vec{A} = (A_1, A_2, A_3)$ being the gauge potential of the Dirac magnetic monopole. Mathematically $\nabla_a = \partial_a + \omega_a$ where $\omega = \omega_a dx_a$ has been previously identified with the Levi-Civita spin connection form of the cylindrical metric

$$ds^2 = \frac{1}{r^2}(dx_1^2 + dx_2^2 + dx_3^2)$$

¹The Dirac quantization condition becomes $\frac{\mu}{\hbar}$ is a half integer.

²Remark that \hat{h} is dimensionless and is expressed in terms of dimensionless quantities. It is \hat{h} (not \hat{H}) that will be generalized later.

³The case $\mu = 0$ corresponds to the Kepler problem.

on the punctured 3-space, see Ref. [4] for the details.

The MICZ-Kepler problems exist in higher dimensions just as the Kepler problem does, and that is a main observation here. In fact, the existence in dimension five has been previously observed [5]; however, the existence in all dimensions greater than two, though very straightforward from a canonical geometric point of view, was probably not expected by the community. This overlook is very likely due to a general belief in the literature: the existence of Dirac monopoles and its five dimensional analogue (the Yang monopoles [6]) has to do with the existence of the division algebras or Hopf bundles.

In section 2, we construct the MICZ-Kepler problems in all dimensions and then state the main results. The construction is geometric and canonical. A key ingredient in the construction is the higher dimensional generalization of the Dirac monopoles — a canonical geometric object that has been used in Ref. [7]. In section 3, we first introduce the explicit formulas for the gauge potential of these generalized Dirac monopoles; then we list and prove some crucial identities necessary for the exhibition of the extra large hidden dynamical symmetry. In section 4, we introduce the angular momentum and Runge-Lenz vector for our MICZ-Kepler problems, and derive the symmetry algebra. In section 5, we obtain the energy spectrum and the energy eigenspaces for bound states by using Painlevé analysis plus representation theory, and then show that the Hilbert space of bound states has a hidden dynamical $\text{Spin}(D+1)$ -symmetry for a D -dimensional MICZ-Kepler problem even though in general the Runge-Lenz vector fails to be conserved when D is even.

Remark that the MICZ-Kepler problems in higher dimensions constructed here are based on modern geometry, but they are solved by classical analytic method with the help of the representation theory for Lie groups. The solution of these new MICZ-Kepler problems can in principle be solved by the modern geometric quantization approach pioneered by Simms [8], Mladenov and Tsanov [9, 10], but that will be reserved for the future for the following reasons: 1) the primary objective of this paper is to inform the experts in the fields that the MICZ-Kepler problems do exist in higher dimensions, 2) the classical analytic approach is more elementary and easier to understand, 3) the modern geometric quantization approach is a bit more involved and deserves an independent research.

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2 The main results

From the physics point of view, a MICZ Kepler problem is obtained from the Kepler problem by adding a suitable background magnetic field, while at the same time making a suitable adjustment to the scalar Coulomb potential so that the problem is still integrable. The background magnetic field is just the spin connection⁴ of the cylindrical metric on the configuration space that we have mentioned in the introduction. The configuration space is the punctured Euclidean space. With this in mind, we are now ready to give the detailed presentation of our generalized MICZ Kepler problems.

Let $D \geq 3$ be an integer, \mathbb{R}_*^D be the punctured D -space, i.e., \mathbb{R}^D with the origin removed. Let ds^2 be the cylindrical metric on \mathbb{R}_*^D . Then (\mathbb{R}_*^D, ds^2) is the product of the straight line \mathbb{R} with the round sphere S^{D-1} . When D is odd, we let \mathcal{S}_\pm be the positive/negative spinor bundle of (\mathbb{R}_*^D, ds^2) , and when D is even, we let \mathcal{S} be the spinor bundle of (\mathbb{R}_*^D, ds^2) . Note that, these bundles correspond to the fundamental spin representations \mathfrak{s}_\pm of $\mathfrak{so}(\text{even})$ and \mathfrak{s} of $\mathfrak{so}(\text{odd})$ respectively.

The above spinor bundles come with a natural $\text{SO}(D)$ invariant connection — the Levi-Civita spin connection of (\mathbb{R}_*^D, ds^2) . As a result, the Young product of I copies of these bundles, denoted by \mathcal{S}_+^I , \mathcal{S}_-^I (when D is odd) and \mathcal{S}^I (when D is even) respectively, come with a natural connection, too.

For the sake of notational sanity, from here on, when D is odd and μ is a half integer, we rewrite $\mathcal{S}_+^{2\mu}$ as $\mathcal{S}^{2\mu}$ if $\mu \geq 0$ and rewrite $\mathcal{S}_-^{2\mu}$ as $\mathcal{S}^{2\mu}$ if $\mu \leq 0$; moreover, we adopt this convention for $\mu = 0$: \mathcal{S}^0 is the product complex line bundle with the product connection. When D is odd, $\mathcal{S}^{2\mu}$ is the product complex line bundle with the product connection in the case $\mu = 0$, and is the fundamental spinor bundle \mathcal{S} in the case $\mu = 1/2$.

Note that $\mathcal{S}^{2\mu}$ is our analogue of the Dirac monopole with magnetic charge μ , and its corresponding representation of $\mathfrak{so}(D-1)$ will be denoted by $\mathfrak{s}^{2\mu}$. We are now ready to present our definitions.

Definition 1. Let $n \geq 1$ be an integer, μ a half integer. The $(2n+1)$ -dimensional MICZ-Kepler problem with magnetic charge μ is defined to be the quantum mechanical system on \mathbb{R}_*^{2n+1} for which the wave-functions are sections of $\mathcal{S}^{2\mu}$, and the hamiltonian is

$$\hat{h} = -\frac{1}{2}\Delta_\mu + \frac{(n-1)|\mu| + \mu^2}{2r^2} - \frac{1}{r} \quad (3)$$

where Δ_μ is the Laplace operator twisted by $\mathcal{S}^{2\mu}$.

Definition 2. Let $n > 1$ be an integer, $\mu = 0$ or $1/2$. The $2n$ -dimensional MICZ-Kepler problem with magnetic charge μ is defined to be the quantum mechanical system on \mathbb{R}_*^{2n} for which the wave-functions are sections of $\mathcal{S}^{2\mu}$, and

⁴For readers without sufficient background in modern geometry, just take our explicit formulas for gauge potential in Eq. (5) for granted.

the hamiltonian is

$$\hat{h} = -\frac{1}{2}\Delta_\mu + \frac{(n-1)\mu}{2r^2} - \frac{1}{r} \quad (4)$$

where Δ_μ is the Laplace operator twisted by $\mathcal{S}^{2\mu}$.

Note that we require $\mu = 0$ or $1/2$ in the even dimensional case. There is both an analytic and an algebraic reason for this requirement, which shall be pointed out in appropriate places. Remark also that, upon a choice of a local gauge, the background magnetic potential A_α can be explicitly written down, then $\Delta_\mu = \sum_\alpha (\partial_\alpha + iA_\alpha)^2$ can be explicitly written down, too. We are now ready to state our main results.

Theorem 1. *Let $n \geq 1$ be an integer and μ be a half integer. For the $(2n+1)$ -dimensional MICZ-Kepler problem with magnetic charge μ , the following statements are true:*

1) *The negative energy spectrum is*

$$E_I = -\frac{1/2}{(I+n+|\mu|)^2}$$

where $I = 0, 1, 2, \dots$;

2) *The Hilbert space \mathcal{H} of negative-energy states admits a linear $\text{Spin}(2n+2)$ -action under which there is a decomposition*

$$\mathcal{H} = \hat{\bigoplus}_{I=0}^{\infty} \mathcal{H}_I$$

where \mathcal{H}_I is a model for the irreducible $\text{Spin}(2n+2)$ -representation with highest weight $(I+|\mu|, |\mu|, \dots, |\mu|, \mu)$;

3) *$\text{Spin}(2n+1, 1)$ acts linearly on the positive-energy states and $\text{Spin}(2n+1) \ltimes \mathbb{R}^{2n+1}$ acts linearly on the zero-energy states;*

4) *The linear action in either part 2) or part 3) extends the manifest linear action of $\text{Spin}(2n+1)$, and \mathcal{H}_I in part 2) is the energy eigenspace with eigenvalue E_I in part 1).*

Theorem 2. *Let $n > 1$ be an integer and $\mu = 0$ or $1/2$. For the $2n$ -dimensional MICZ-Kepler problem with magnetic charge μ , the following statements are true:*

1) *The negative energy spectrum is*

$$E_I = -\frac{1/2}{(I+n+\mu-\frac{1}{2})^2}$$

where $I = 0, 1, 2, \dots$;

2) *The Hilbert space \mathcal{H} of negative-energy states admits a linear $\text{Spin}(2n+1)$ -action under which there is a decomposition*

$$\mathcal{H} = \hat{\bigoplus}_{I=0}^{\infty} \mathcal{H}_I$$

where \mathcal{H}_I is a model for the irreducible $\text{Spin}(2n+1)$ -representation with highest weight $(I + \mu, \mu, \dots, \mu)$;

3) $\text{Spin}(2n, 1)$ acts linearly on the positive-energy states and $\text{Spin}(2n) \rtimes \mathbb{R}^{2n}$ acts linearly on the zero-energy states;

4) The linear action in part 2) extends the manifest linear action of $\text{Spin}(2n)$, and \mathcal{H}_I in part 2) is the energy eigenspace with eigenvalue E_I in part 1).

Remark that, based on the analysis done in later sections, we know that bound eigen-states are always the ones with negative energy eigenvalues.

3 Generalized Dirac monopoles

We write $\vec{r} = (x_1, x_2, \dots, x_{D-1}, x_0)$ for a point in \mathbb{R}^D and r for the length of \vec{r} . The small Greek letters μ, ν , etc run from 0 to $D-1$ and the small Latin letters a, b etc run from 1 to $D-1$. We use the Einstein convention: the repeated index is always summed up.

To do computations, we just need to choose a gauge on \mathbb{R}^D minus the negative 0-th axis and then write down the gauge potential explicitly. We have done that before in Eq. (10) of Ref. [4]. Note that, if we use the rectangular coordinates $\vec{r} = (\vec{x}, x_0)$, then the gauge potential $A = A_\mu dx_\mu$ from Eq. (10) of Ref. [4] can be written as⁵

$$A_0 = 0, \quad A_b = -\frac{1}{r(r+x_0)} x_a \gamma_{ab} \quad (5)$$

where $\gamma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$ with γ_a being the “gamma matrix” for physicists. Note that $\gamma_a = ie_a$ with \vec{e}_a being the element in the Clifford algebra that corresponds to the a -th standard coordinate vector of \mathbb{R}^{D-1} .

It is straightforward to calculate the gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and get

$$F_{0b} = \frac{1}{r^3} x_a \gamma_{ab} \quad (6)$$

$$F_{ab} = -\frac{2\gamma_{ab}}{r(r+x_0)} + \frac{1}{r^2(r+x_0)^2} \cdot \left(\left(2 + \frac{x_0}{r}\right) x_c (x_a \gamma_{cb} - x_b \gamma_{ca}) + i x_d x_c [\gamma_{da}, \gamma_{cb}] \right) \quad (7)$$

The following lemma is crucially used when we check the dynamical symmetry of our models.

Lemma 1. *For the gauge potential defined in Eq. (5), we have*

⁵In Ref. [4] we only consider the case D is odd — the topological nontrivial case, but the basic construction there is valid in any dimension, see appendix A in Ref. [7].

1) Let $\nabla_\alpha = \partial_\alpha + iA_\alpha$, then the following identities are valid in any representation:

$$F_{\mu\nu}F^{\mu\nu} = \frac{2}{r^4}c_2 \quad \text{where } c_2 = c_2[\text{so}(D-1)] = \frac{1}{2}\gamma_{ab}\gamma_{ab} \quad (8)$$

$$[\nabla_\kappa, F_{\mu\nu}] = \frac{1}{r^2}(x_\mu F_{\nu\kappa} + x_\nu F_{\kappa\mu} - 2x_\kappa F_{\mu\nu}) \quad (9)$$

$$x_\mu A_\mu = 0, \quad x_\mu F_{\mu\nu} = 0, \quad [\nabla_\mu, F_{\mu\nu}] = 0 \quad (10)$$

$$\begin{aligned} r^2[F_{\mu\nu}, F_{\alpha\beta}] + iF_{\mu\beta}\delta_{\alpha\nu} - iF_{\nu\beta}\delta_{\alpha\mu} + iF_{\alpha\mu}\delta_{\beta\nu} - iF_{\alpha\nu}\delta_{\beta\mu} \\ = \frac{i}{r^2}(x_\mu x_\alpha F_{\beta\nu} + x_\mu x_\beta F_{\nu\alpha} - x_\nu x_\alpha F_{\beta\mu} - x_\nu x_\beta F_{\mu\alpha}) \end{aligned} \quad (11)$$

2) When $D = 2n + 1$, identity

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{c_2}{n} \left(\frac{1}{r^2} \delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{r^4} \right) + i(n-1)F_{\alpha\beta} \quad (12)$$

holds in the irreducible representation $\mathbf{s}^{2\mu}$ of $\text{so}(2n)$ whose highest weight is of the form $(|\mu|, \dots, |\mu|, \mu)$.

3) When $D = 2n$, identity

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{n-1}{2} \left(\frac{1}{r^2} \delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{r^4} \right) + i(n - \frac{3}{2})F_{\alpha\beta} \quad (13)$$

holds in the fundamental spin representation \mathbf{s} of $\text{so}(2n-1)$.

One can show that the Eq. (13) is valid only when $\mu = 0$ or $1/2$. That is the algebraic reason for requiring $\mu = 0$ or $1/2$.

3.1 Proof of Lemma 1

The verification of these identities is just a direct and lengthy calculation. However, if we exploit the symmetry, we just need to check the identities at point $\vec{r}_0 = (0, \dots, 0, r)$, a much easier task. For example, since

$$A_\mu = 0, \quad F_{0a} = 0, \quad F_{ab} = -\frac{1}{r^2}\gamma_{ab} \quad (14)$$

at \vec{r}_0 , identity (8) is obvious.

Proof of part 1). We have just remarked that identity (8) is obvious. Also,

$$x_\mu F_{\mu\nu}|_{\vec{r}_0} = x_0 F_{0\nu}|_{\vec{r}_0} = 0.$$

It is also easy to see that $x_\mu A_\mu|_{\vec{r}_0} = 0$ and

$$[\nabla_\mu, F_{\mu\nu}]|_{\vec{r}_0} = \partial_\mu F_{\mu\nu}|_{\vec{r}_0} = 0.$$

Therefore, identity (10) is checked.

To check identity (9), first we assume $\mu = 0$, $\nu = b$, then we need to check that

$$\partial_\kappa F_{0b} = \frac{1}{r} F_{b\kappa}$$

at \vec{r}_0 , and that can be easily seen to be true whether $\kappa = 0$ or a . Next we assume that $\mu = a$ and $\nu = b$, then we need to check that

$$\partial_\kappa F_{ab} = -\frac{2}{r^2} x_\kappa F_{ab}$$

at \vec{r}_0 , and that can be easily verified, too.

We divide the checking of identity (11) at \vec{r}_0 into two cases: 1) one of indices is zero, easy to check; 2) none of the indices is zero, then the identity becomes

$$-[\gamma_{ab}, \gamma_{cd}] = -i\gamma_{ad}\delta_{bc} + i\gamma_{bd}\delta_{ac} - i\gamma_{ca}\delta_{bd} + i\gamma_{cb}\delta_{ad} \quad (15)$$

which is of course true because γ_{ab} 's are the generators of $\mathfrak{so}(D-1)$.

Proof of part 2). Write

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{r^2}{2} \{F_{\lambda\alpha}, F_{\lambda\beta}\} + \frac{r^2}{2} [F_{\lambda\alpha}, F_{\lambda\beta}].$$

Using identity (12), we have

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{r^2}{2} \{F_{\lambda\alpha}, F_{\lambda\beta}\} + i(n-1)F_{\alpha\beta}.$$

Therefore, by checking at \vec{r}_0 , we just need to verify that identity

$$\sum_k \{\gamma_{ki}, \gamma_{kj}\} = \frac{\delta_{ij}}{n} \sum_{a,b} (\gamma_{ab})^2 \quad (16)$$

holds in the irreducible representation $\mathfrak{s}^{2\mu}$ of $\mathfrak{so}(2n)$ whose highest weight is of the form $(|\mu|, \dots, |\mu|, \mu)$. Since this checking is a bit involved, we do it in the appendix.

Proof of part 3). Write

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{r^2}{2} \{F_{\lambda\alpha}, F_{\lambda\beta}\} + \frac{r^2}{2} [F_{\lambda\alpha}, F_{\lambda\beta}].$$

Using identity (12), we have

$$r^2 F_{\lambda\alpha} F_{\lambda\beta} = \frac{r^2}{2} \{F_{\lambda\alpha}, F_{\lambda\beta}\} + i(n - \frac{3}{2})F_{\alpha\beta}.$$

Therefore, by checking at \vec{r}_0 , we just need to verify that identity

$$\sum_k \{\gamma_{ki}, \gamma_{kj}\} = (n-1)\delta_{ij} \quad (17)$$

holds in the spin representation \mathfrak{s} of $\mathfrak{so}(2n-1)$, but this is easy to check by using the Clifford algebra.

4 The hidden dynamical symmetry

To exhibit the dynamical symmetry for our MICZ-Kepler problems, as usual, we introduce the angular momentum tensor

$$\hat{L}_{\alpha\beta} = -i(x_\alpha \nabla_\beta - x_\beta \nabla_\alpha) + r^2 F_{\alpha\beta} \quad (18)$$

and the Runge–Lenz vector

$$\hat{L}_\beta = -\frac{i}{2} \left(\nabla_\alpha \hat{L}_{\alpha\beta} + \hat{L}_{\alpha\beta} \nabla_\alpha \right) + \frac{x_\beta}{r} \quad (19)$$

With the help of the identities stated in lemma 1, a lengthy calculation yields the following commutation relations:

$[\hat{L}_{\mu\nu}, \hat{h}]$	$= 0$	(20)
$[\hat{L}_{\mu\nu}, \hat{L}_{\alpha\beta}]$	$= i\delta_{\mu\alpha}\hat{L}_{\nu\beta} - i\delta_{\nu\alpha}\hat{L}_{\mu\beta} - i\delta_{\mu\beta}\hat{L}_{\nu\alpha} + i\delta_{\nu\beta}\hat{L}_{\mu\alpha}$	
$[\hat{L}_{\mu\nu}, \hat{L}_\lambda]$	$= i\delta_{\mu\lambda}\hat{L}_\nu - i\delta_{\nu\lambda}\hat{L}_\mu$	
$[\hat{L}_\mu, \hat{h}]$	$= 0$	
$[\hat{L}_\mu, \hat{L}_\nu]$	$= -2i\hat{h}\hat{L}_{\mu\nu}.$	

On the Hilbert space of negative-energy states, we can introduce \hat{J}_{MN} where the capital Latin letters M, N run from 0 to D :

$$\hat{J}_{MN} = \begin{cases} \hat{L}_{\mu\nu} & \text{if } M = \mu, N = \nu \\ (-2\hat{h})^{-\frac{1}{2}} \hat{L}_\mu & \text{if } M = \mu, N = D \\ -(-2\hat{h})^{-\frac{1}{2}} \hat{L}_\nu & \text{if } M = D, N = \nu \\ 0 & \text{if } M = N \end{cases}$$

Then the commutation relation (20) says that a D -dimensional MICZ-Kepler problem has a dynamical $\text{SO}(D+1)$ -symmetry on the Hilbert space of negative-energy states. Actually, the dynamical symmetry group should be $\text{Spin}(D+1)$, rather than $\text{SO}(D+1)$. It is also clear that a D -dimensional MICZ-Kepler problem has a dynamical $\text{Spin}(D,1)$ -symmetry on the positive-energy states and a dynamical $\text{Spin}(D) \rtimes \mathbb{R}^D$ -symmetry on the zero-energy states.

It also follows from (20) that \hat{h} must be in the center of Lie algebra $\mathfrak{so}(D+1)$; in fact, it is a function of the quadratic Casimir operator of $\mathfrak{so}(D+1)$:

$$\hat{h} = -\frac{1/2}{c_2[\mathfrak{so}(D+1)] + (\frac{D-1}{2})^2 - \bar{c}_2} \quad (21)$$

where \bar{c}_2 is the value of $c_2[\mathfrak{so}(D-1)]$ in representation $\mathbf{s}^{2\mu}$.

To prove Eq. (21), we first note that $\hat{L}_\mu = (-2\hat{h})^{\frac{1}{2}} \hat{J}_{\mu D}$, so $\hat{L}_\mu \hat{L}_\mu = -2\hat{h} \sum_\mu \hat{J}_{\mu D} \hat{J}_{\mu D}$. On the other hand, based on the definition of L_μ given in Eq. (19), a direct computation yields

$$\hat{L}_\mu \hat{L}_\mu = 1 + \left(\frac{1}{2}(D-1)^2 - 2\bar{c}_2 + \hat{J}_{\mu\nu} \hat{J}_{\mu\nu} \right) \hat{h}.$$

Therefore,

$$1 + \left(\frac{1}{2}(D-1)^2 - 2\bar{c}_2 + \hat{J}_{MN}\hat{J}_{MN} \right) \hat{h} = 0,$$

then we have Eq. (21).

It is clear now that in order to determine the spectrum of \hat{h} , we just need to find out which irreducible representation of $\text{Spin}(D+1)$ enters into the Hilbert space \mathcal{H} of negative-energy states. However, we shall find the discrete spectrum by solving the Schrödinger equation directly and then figure out the decomposition of the Hilbert space of negative-energy states into the irreducible representations of $\text{Spin}(D+1)$ via representation theory.

5 The spectrum analysis

The Schrödinger equation for the stationary states, in terms of the polar coordinates, is

$$\left(-\frac{1}{2r^{D-1}}\partial_r r^{D-1}\partial_r + \frac{c_2[\text{so}(D)] - \bar{c}_2 + \delta_D}{2r^2} - \frac{1}{r} \right) \psi = E\psi \quad (22)$$

where E is the energy, $c_2[\text{so}(D)] = \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}_{\mu\nu}$, and δ_D is equal to $(n-1)\mu$ if $D = 2n$ and is equal to $(n-1)|\mu| + \mu^2$ if $D = 2n+1$. Under the action of $\text{Spin}(D)$, the Hilbert space of negative-energy states splits into the direct sum of irreducible components. These irreducible components are essentially labeled by a nonnegative integer l , and we shall be able to see that shortly. On the irreducible component labeled by l , the Schrödinger equation becomes an equation for the radial part:

$$\left(-\frac{1}{2r^{D-1}}\partial_r r^{D-1}\partial_r + \frac{c_2[l] - \bar{c}_2 + \delta_D}{2r^2} - \frac{1}{r} \right) R_{kl} = E_{kl}R_{kl} \quad (23)$$

where $c_2[l]$ is the value of the quadratic Casimir operator of $\text{so}(D)$ in the irreducible component labeled by l , and the additional label k is introduced for the purpose of listing the radial eigenfunctions, just as in the Kepler problem.

Let $E_{kl} = -\frac{1}{2}\lambda_{kl}^2$ and $R_{kl}(r) = e^{-\lambda_{kl}r}u_{kl}$, then the preceding radial Schrödinger equation becomes

$$\left(-\frac{1}{2r^{D-1}}\partial_r r^{D-1}\partial_r + \lambda_{kl}\frac{1}{r^{\frac{D-1}{2}}}\partial_r r^{\frac{D-1}{2}} + \frac{c_2[l] - \bar{c}_2 + \delta_D}{2r^2} - \frac{1}{r} \right) u_{kl} = 0 \quad (24)$$

Let $y_{kl} = r^{\frac{D-1}{2}}u_{kl}$, then the above equation becomes

$$\left(\frac{d^2}{dr^2} - 2\lambda_{kl}\frac{d}{dr} + \left[\frac{2}{r} - \frac{c_2[l] - \bar{c}_2 + \delta_D + \frac{(D-1)(D-3)}{4}}{r^2} \right] \right) y_{kl}(r) = 0 \quad (25)$$

Assume that $y_{kl}(r) \rightarrow r^s$ as $r \rightarrow 0^+$, then we must have the following indicial equation:

$$\boxed{s(s-1) = c_2[l] - \bar{c}_2 + \delta_D + \frac{(D-1)(D-3)}{4}}. \quad (26)$$

The further analysis is divided into two cases: 1) D is odd, 2) D is even.

5.1 The odd dimensional cases

Let $D = 2n + 1$. Let $L^2(\mathcal{S}^{2\mu}|_{\mathbb{S}^{2n}})$ be the L^2 -sections of vector bundle $\mathcal{S}^{2\mu}$ restricted to the unit sphere \mathbb{S}^{2n} . From the representation theory, we know that

$$L^2(\mathcal{S}^{2\mu}|_{\mathbb{S}^{2n}}) = \hat{\bigoplus}_{l \geq 0} \mathcal{R}_l \quad (27)$$

where \mathcal{R}_l is the irreducible representation space of $\text{Spin}(2n+1)$ with highest weight $(l + |\mu|, |\mu|, \dots, |\mu|)$. It is then clear that the Hilbert spaces of bound states is

$$\mathcal{H} = \hat{\bigoplus}_{l \geq 0} \mathcal{H}_l \quad (28)$$

with \mathcal{H}_l being a subspace of $L^2(\mathbb{R}_+, r^{2n} dr) \otimes \mathcal{R}_l$. Here $L^2(\mathbb{R}_+, r^{2n} dr)$ is the L^2 -space of complex-valued functions on half-line \mathbb{R}_+ with measure $r^{2n} dr$.

The value of the quadratic Casimir operator of $\mathfrak{so}(2n)$ on representation $\mathfrak{s}^{2\mu}$ is

$$\bar{c}_2 = n\mu^2 + n(n-1)|\mu|.$$

The value of the quadratic Casimir operator of $\mathfrak{so}(2n+1)$ on \mathcal{R}_l is

$$c_2[l] = l^2 + 2l(n + |\mu| - \frac{1}{2}) + n\mu^2 + n^2|\mu|.$$

Plugging the values for \bar{c}_2 and $c_2[l]$ into Eq. (26), we get

$$s(s-1) = (l + n + |\mu|)(l + n + |\mu| - 1).$$

Therefore, $s = l + n + |\mu|$ or $s = 1 - l - n - |\mu|$. The solution $s = 1 - l - n - |\mu|$ must be rejected; otherwise, the wave-functions cannot be square integrable near $r = 0$. Just as in solving the hydrogen atom problem, with $s = l + n + |\mu|$, we continue the analysis by setting

$$y_{kl} = r^s \sum_{m=0}^{\infty} a_m r^m$$

with $a_0 = 1$ and then get the recursive relation: for $m \geq 1$, one has

$$a_m ((m+s)(m+s-1) - s(s-1)) = (1 - \lambda_{kl}(m+s-1)) a_{m-1}. \quad (29)$$

As it has been demonstrated in Ref. [11], the power series solution must be a polynomial solution; otherwise, the wave-function will not be square integrable for r near infinity. Therefore, we must have

$$\lambda_{kl} = \frac{1}{k+s-1} = \frac{1}{k+l+n+|\mu|-1}$$

and that leads to the energy spectrum

$$E_{kl} = -\frac{1/2}{(k+l+n+|\mu|-1)^2} \quad (30)$$

where k must be a positive integer, and an orthogonal decomposition

$$\mathcal{H}_l = \bigoplus_{k=1}^{\infty} \mathcal{H}_{kl}$$

with each of \mathcal{H}_{kl} being isomorphic to \mathcal{R}_l as $\text{Spin}(2n+1)$ -modules. Therefore, in view of Eq. (28), we have an orthogonal decomposition of \mathcal{H} into energy eigen-states:

$$\mathcal{H} = \bigoplus_{I=0}^{\infty} \mathcal{H}_I \quad (31)$$

where

$$\mathcal{H}_I = \bigoplus_{k+l=I+1} \mathcal{H}_{kl}.$$

Since linear action of $\text{Spin}(2n+2)$ on \mathcal{H} commutes with the hamiltonian, this linear action must leave the energy eigen-states \mathcal{H}_I invariant. On the other hand, from representation theory, as a $\text{Spin}(2n+1)$ -module, being isomorphic to

$$\bigoplus_{l=0}^I \mathcal{R}_l,$$

\mathcal{H}_I must be the irreducible representation of $\text{Spin}(2n+2)$ with highest weight $(I+|\mu|, |\mu|, \dots, |\mu|, \pm|\mu|)$. As a consistency check, one can see that Eq. (21) yields

$$E_I = -\frac{1/2}{(I+n+|\mu|)^2} \quad (32)$$

on such representation, in complete agreement with Eq. (30) because $I = k+l-1$. One can show that, as a $\text{Spin}(2n+2)$ -module, \mathcal{H}_I has the highest weight $(I+|\mu|, |\mu|, \dots, |\mu|, \mu)$.

In summary, the energy spectrum is

$$E_I = -\frac{1/2}{(I+n+|\mu|)^2} \quad (33)$$

where $I = 0, 1, 2, \dots$; and \mathcal{H} furnishes a representation for $\text{Spin}(2n+2)$ and has the following decomposition into energy eigenstates:

$$\mathcal{H} = \hat{\bigoplus}_{I=0}^{\infty} \mathcal{H}_I \quad (34)$$

where \mathcal{H}_I is the irreducible component of \mathcal{H} with highest weight $(I + |\mu|, |\mu|, \dots, |\mu|, \mu)$.

5.2 The even dimensional cases

Let $D = 2n$. Let $L^2(\mathcal{S}^{2\mu}|_{\mathbb{S}^{2n-1}})$ be the L^2 -sections of vector bundle $\mathcal{S}^{2\mu}$ restricted to the unit sphere \mathbb{S}^{2n-1} . From the representation theory, we know that⁶

$$L^2(\mathcal{S}^{2\mu}|_{\mathbb{S}^{2n-1}}) = \begin{cases} \hat{\bigoplus}_{l \geq 0} (\mathcal{R}_l^+ \oplus \mathcal{R}_l^-) & \text{if } \mu = 1/2 \\ \hat{\bigoplus}_{l \geq 0} \mathcal{R}_l^0 & \text{if } \mu = 0 \end{cases} \quad (35)$$

where \mathcal{R}_l^\pm is the irreducible representation of $\text{Spin}(2n)$ with highest weight $(l+1/2, 1/2, \dots, 1/2, \pm 1/2)$ and \mathcal{R}_l^0 is the irreducible representation of $\text{Spin}(2n)$ with highest weight $(l, 0, \dots, 0)$. It is then clear that the Hilbert spaces of bound states is

$$\mathcal{H} = \begin{cases} \hat{\bigoplus}_{l \geq 0} (\mathcal{H}_l^+ \oplus \mathcal{H}_l^-) & \text{if } \mu = 1/2 \\ \hat{\bigoplus}_{l \geq 0} \mathcal{H}_l^0 & \text{if } \mu = 0 \end{cases} \quad (36)$$

with \mathcal{H}_l^σ being a subspace of $L^2(\mathbb{R}_+, r^{2n-1} dr) \otimes \mathcal{R}_l^\sigma$. Here $L^2(\mathbb{R}_+, r^{2n-1} dr)$ is the L^2 -space of complex-valued functions on half-line \mathbb{R}_+ with measure $r^{2n-1} dr$.

The value of the quadratic Casimir operator of $\mathfrak{so}(2n-1)$ on representation $\mathcal{S}^{2\mu}$ is

$$\bar{c}_2 = (n-1)\mu^2 + (n-1)^2\mu.$$

The value of the quadratic Casimir operator of $\mathfrak{so}(2n)$ on \mathcal{R}_l^σ is

$$c_2[l] = l^2 + 2l(n + \mu - 1) + n\mu^2 + (n^2 - n)\mu. \quad (37)$$

Plugging the values for \bar{c}_2 and $c_2[l]$ into Eq. (26), we get

$$s(s-1) = (l+n+\mu - \frac{1}{2})(l+n+\mu - \frac{3}{2}).$$

Therefore, $s = l+n+\mu - \frac{1}{2}$ or $s = \frac{3}{2} - l - n - \mu$. The solution $s = \frac{3}{2} - l - n - \mu$ must be rejected; otherwise, the wave-functions cannot be square integrable near

⁶For a fixed l , there are $(2\mu+1)$ many of \mathcal{R}_l 's. When $\mu > 1/2$, Eq. (37) below is no longer valid for some \mathcal{R}_l and the subsequent analysis fails. That is the analytic reason for requiring $\mu = 0$ or $1/2$.

$r = 0$. Just as in solving the hydrogen atom problem, with $s = l + n + \mu - \frac{1}{2}$, we continue the analysis by setting

$$y_{kl} = r^s \sum_{m=0}^{\infty} a_m r^m$$

with $a_0 = 1$ and then get the recursive relation: for $m \geq 1$, one has

$$a_m ((m+s)(m+s-1) - s(s-1)) = (1 - \lambda_{kl}(m+s-1)) a_{m-1}. \quad (38)$$

As it has been demonstrated in Ref. [11], the power series solution must be a polynomial solution; otherwise, the wave-function will not be square integrable for r near infinity. Therefore, we must have

$$\lambda_{kl} = \frac{1}{k+s-1} = \frac{1}{k+l+n+\mu-\frac{3}{2}}$$

and that leads to the energy spectrum

$$E_{kl} = -\frac{1/2}{(k+l+n+\mu-\frac{3}{2})^2} \quad (39)$$

where k must be a positive integer, and an orthogonal decomposition

$$\mathcal{H}_l^\sigma = \hat{\bigoplus}_{k=1}^{\infty} \mathcal{H}_{kl}^\sigma$$

with each \mathcal{H}_{kl}^σ being isomorphic to \mathcal{R}_l^σ as $\text{Spin}(2n)$ -modules. Therefore, in view of Eq. (36), we have an orthogonal decomposition of \mathcal{H} into energy eigen-states:

$$\mathcal{H} = \hat{\bigoplus}_{I=0}^{\infty} \mathcal{H}_I \quad (40)$$

where

$$\mathcal{H}_I = \begin{cases} \bigoplus_{k+l=I+1} (\mathcal{H}_{kl}^+ \oplus \mathcal{H}_{kl}^-) & \text{if } \mu = 1/2 \\ \bigoplus_{k+l=I+1} \mathcal{H}_{kl}^0 & \text{if } \mu = 0. \end{cases}$$

Note that \mathcal{H}_I is isomorphic to

$$\begin{cases} \bigoplus_{l=0}^I (\mathcal{R}_l^+ \oplus \mathcal{R}_l^-) & \text{if } \mu = 1/2 \\ \bigoplus_{l=0}^I \mathcal{R}_l^0 & \text{if } \mu = 0 \end{cases}$$

as a $\text{Spin}(2n)$ -module, from the representation theory, the manifest $\text{Spin}(2n)$ linear action on \mathcal{H}_I can be extended to a linear action of $\text{Spin}(2n+1)$ such that \mathcal{H}_I is the irreducible representation of $\text{Spin}(2n+1)$ with highest weight

$(I + \mu, \mu, \dots, \mu)$ and Eq. (21) is valid. As a consistency check, one can see that Eq. (21) yields

$$E_I = -\frac{1/2}{(I + n + \mu - \frac{1}{2})^2} \quad (41)$$

on such representation, in complete agreement with Eq. (39) because $I = k + l - 1$.

In summary, the energy spectrum is

$$E_I = -\frac{1/2}{(I + n + \mu - \frac{1}{2})^2} \quad (42)$$

where $I = 0, 1, 2, \dots$; and \mathcal{H} furnishes a representation for $\text{Spin}(2n + 1)$ and has the following decomposition into energy eigenstates:

$$\mathcal{H} = \hat{\bigoplus}_{I=0}^{\infty} \mathcal{H}_I \quad (43)$$

where \mathcal{H}_I is the irreducible component of \mathcal{H} with highest weight $(I + \mu, \mu, \dots, \mu)$.

A Proof of the remaining part of Lemma 1

The case $n = 1$ is trivial. So we assume that $n \geq 2$. To prove identity (16), we first note that we just need to prove that identities

$$\sum_k (\gamma_{1,k})^2 = \frac{1}{n} c_2 \quad (44)$$

and

$$\sum_k \{\gamma_{1,k}, \gamma_{2,k}\} = 0 \quad (45)$$

hold in the representation $\mathbf{s}_+^{2\mu}$ for any non-negative half integer μ .

To continue, a digression on Lie algebra $\mathfrak{so}(2n)$ is needed. Recall that the root space of $\mathfrak{so}(2n)$ is \mathbb{R}^n . Let e^i be the vector in \mathbb{R}^n whose i -th entry is 1 and all other entries are zero. The positive roots are $e^i \pm e^j$ with $1 \leq i < j \leq n$. The simple roots are $\alpha^i = e^i - e^{i+1}$, $i = 1$ to $n - 1$ and $\alpha^n = e^{n-1} + e^n$. For the Cartan basis, we make the following choice: The commuting generators in the Cartan subalgebra are taken to be

$$H_j = -\gamma_{2j-1,2j} \quad j = 1 \text{ to } n,$$

and the E generators are taken to be

$$E_{\eta e^j + \eta' e^k} = -\frac{1}{2} (\gamma_{2j-1,2k-1} + i\eta\gamma_{2j,2k-1} + i\eta'\gamma_{2j-1,2k} - \eta\eta'\gamma_{2j,2k}) \quad (46)$$

where $j < k$, and $\eta, \eta' \in \{1, -1\}$. Note that, the fact that

$$[E_\alpha, E_\beta] = 0 \quad \text{if } \alpha + \beta \text{ is neither a root nor zero}$$

is frequently used in all subsequent calculations. All of these are standard materials taken from a textbook such as Ref. [12].

Let

$$\mathcal{O} = \sum_{n \geq i \geq 2} E_{-e^1 - e^i} E_{-e^1 + e^i} \quad (47)$$

$$\mathcal{O}_1 = H_1^2 + \frac{1}{2} \sum_{n \geq i \geq 2} (\{E_{-e^1 - e^i}, E_{e^1 + e^i}\} + \{E_{-e^1 + e^i}, E_{e^1 - e^i}\}) \quad (48)$$

By simple computations, we have

$$\sum_k (\gamma_{1,k})^2 = \mathcal{O}_1 + \mathcal{O}^\dagger + \mathcal{O} \quad (49)$$

$$\sum_k \{\gamma_{1,k}, \gamma_{2,k}\} = \frac{2}{i} (\mathcal{O}^\dagger - \mathcal{O}) \quad (50)$$

It is then clear from the above calculations that identities (44) and (45) are valid modulo the following claim:

Claim 1. *Let $|\Lambda\rangle$ be an element of the $\mathfrak{so}(2n)$ -module $\mathfrak{s}_+^{2\mu}$. Then*

$$\mathcal{O}|\Lambda\rangle = 0, \quad (51)$$

$$\mathcal{O}^\dagger|\Lambda\rangle = 0, \quad (52)$$

$$\mathcal{O}_1|\Lambda\rangle = \mu(n + \mu - 1)|\Lambda\rangle \quad (53)$$

$$= \frac{1}{n} c_2 |\Lambda\rangle. \quad (54)$$

Proof of the claim. We first remark that, among the four equalities in the claim, we just need to prove the first and the third, that is because the 2nd is a consequence of the first and the last is true because $c_2 = n\mu(n + \mu - 1)$. We also remark that we may assume that $|\Lambda\rangle$ is a state created from $|\mu, \dots, \mu\rangle$ by applying a bunch of lowering operators of the form $E_{-\alpha_j}$'s, that is because a general state is always a linear combination of states of this kind.

Next, we observe that

$$E_{-e^1 + e^i} |\mu, \dots, \mu\rangle = 0; \quad (55)$$

consequently, $\mathcal{O}|\mu, \dots, \mu\rangle = 0$. This observation can be shown by the following trick: Let $e = E_{e^1 - e^i}$, $f = E_{-e^1 + e^i}$, $h = H_1 - H_i$, then

$$\begin{aligned} [h, e] &= 2e, \\ [h, f] &= -2f, \end{aligned}$$

$$[e, f] = h.$$

I.e., $\{e, f, h\}$ forms the standard Cartan basis for $\mathfrak{su}(2)$. It follows from the following computation

$$\|f|\mu, \dots, \mu\rangle\|^2 = \langle \mu, \dots, \mu | ef | \mu, \dots, \mu \rangle = \langle \mu, \dots, \mu | h + fe | \mu, \dots, \mu \rangle = 0$$

that $f|\mu, \dots, \mu\rangle = 0$. Moreover, when $|\Lambda\rangle = |\mu, \dots, \mu\rangle$, the third identity of the claim is just the consequence of a direct computation. Therefore, the claim is true when $|\Lambda\rangle = |\mu, \dots, \mu\rangle$.

Finally, we need to reduce the general case to the special case discussed in the previous paragraph. Combining the computational fact that

$$[E_{-e^1+e^j}, E_{-e^j+e^{j+1}}] = -iE_{-e^1+e^{j+1}}, [E_{-e^1-e^{j+1}}, E_{-e^j+e^{j+1}}] = iE_{-e^1-e^j} \quad (56)$$

(where $1 < j < n$) and the computational fact that

$$[E_{-e^1+e^{n-1}}, E_{-e^{n-1}-e^n}] = -iE_{-e^1-e^n}, [E_{-e^1+e^n}, E_{-e^{n-1}-e^n}] = iE_{-e^1-e^{n-1}}, \quad (57)$$

one can show that $[\mathcal{O}, E_{-\alpha^j}] = 0$ for any $1 \leq j \leq n$. Since $|\Lambda\rangle$ can be assumed to be a state created from $|\mu, \dots, \mu\rangle$ by applying a bunch of lowering operators of the form $E_{-\alpha^j}$'s, in view of the fact that $\mathcal{O}|\mu, \dots, \mu\rangle = 0$, we have $\mathcal{O}|\Lambda\rangle = 0$.

The reduction to the special case for the third identity is a bit involved. The key is to introduce a series of operators:

$$\begin{aligned} \mathcal{O}_2 &= -2[\mathcal{O}_1, E_{-\alpha^1}] \\ \mathcal{O}_k &= i[\mathcal{O}_{k-1}, E_{-\alpha^{k-1}}] \quad \text{for } 3 \leq k \leq n \\ \mathcal{O}^{n-1} &= -i[\mathcal{O}_{n-1}, E_{-\alpha^n}] \\ \mathcal{O}^{k-1} &= -i[\mathcal{O}^k, E_{-\alpha^k}] \quad \text{for } 1 \leq k \leq n-1 \end{aligned} \quad (58)$$

and then observe that

$$\begin{aligned} [\mathcal{O}_k, E_{-\alpha^j}] &= 0 \quad \text{if } k \neq n-1 \text{ and } j \neq k \\ [\mathcal{O}_{n-1}, E_{-\alpha^j}] &= 0 \quad \text{if } j \neq n-1, n \\ [\mathcal{O}^k, E_{-\alpha^j}] &= 0 \quad \text{if } j \neq k \\ \mathcal{O}^0 &= 4i\mathcal{O} \\ \mathcal{O}_k|\mu, \dots, \mu\rangle &= 0 \quad \text{for } 2 \leq k \leq n \\ \mathcal{O}^k|\mu, \dots, \mu\rangle &= 0 \quad \text{for } 0 \leq k \leq n-1 \end{aligned} \quad (59)$$

With the help of the above equations starting from (58) and ending at (59), an induction argument finishes the proof. Here the induction is done on the number of lowering operators of the form $E_{-\alpha^j}$'s which are used to create state $|\Lambda\rangle$.

We end this appendix with the following conjecture: *identity (16) holds if and only if when the representation is a Young power of \mathbf{s}_+ or \mathbf{s}_-* . For our purpose, we have just proved the “if” part in this appendix.

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